for Hydrogen we have $\tau_p = 3.8 \times 10^{-14}$ sec. at $T = 14.89°K$ and $\tau_p = 5.39 \times 10^{-14}$ sec. at $T = 19.92°K$, while for Argon we have $\tau_p = 4.798 \times 10^{-14}$ sec. at $T = 90.03$ °K.

Combining (20) and (22) we obtain an inverse relationship between τ_p and τ_q, given by

$$\tau_p = \frac{mkT}{\langle K^2 \rangle \tau_q} \tag{23}$$

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ON THE FORMULATION OF THE DEFINING EQUATIONS FOR A PLANE FLOW OF A CONTINUOUS MEDIUM WITH DRY FRICTION

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V. N. NIKOLAEVSKII
(Moscow)

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Construction of the defining equations for the critical equilibrium state of an incompressible, continuous, free-running medium with dry friction [1], presents certain fundamental difficulties. As we know, the lines (areas) of the critical equilibrium state (along the tangents up to which the Coulomb's condition holds), are situated symmetrically, at an angle α to the direction of the highest normal stress and only when the angle of internal friction is equal to zero $\Phi = 0$ (the case of the perfectly plastic body), then the lines of the critical equilibrium coincide ($\alpha = {}^1/_4\pi$) with the lines of the maximum shear rate (the second Saint-Venant condition [2] specifying the plastic state).

If, on the other hand, $\Phi \neq 0$, then the angle between the lines of the critical equilibrium state is equal to $2\alpha = {}^1/_2\pi - \Phi$, i.e. it is different from 90°, and the second Saint-Venant condition no longer holds. Thus, the formal incorporation [3] of the hypothesis that the stress and the rate of strain tensors are collinear in a free-running, incompressible medium with dry friction conradicts the physical model (see e.g. [3]) in which the direction of the largest displacement coincides with the direction of the greatest resistance of the medium.

For this reason the author of [4] proposed the condition of coincidence of only a single slip line (maximum shear rate) with the critical equilibrium line. Other variants of the method of constructing the theory also exist (see e.g. [5]). It should however be noted that the defining relations proposed in [5] do not satisfy the usual invariance conditions [6] for the mechanics of continuous media.

In this paper we introduce a generalized formulation of the critical equilibrium condition, according to which the reaction R of the medium balancing the stresses on the critical equilibrium line (area) makes the angle β with the tangent to this line. Assuming that the force R is collinear with the direction of the maximum shear, we find, that the angle β is equal to the true angle of internal friction Φ_0, while the effective critical equilibrium condition reduces, after redefining the internal friction and cohesion angles, to the usual form of the Coulomb's condition. Thus we arrive at such interpretation of the critical steady state of the free-running medium with dry friction which does not contradict the hypothesis of collinearity of the stress and rate of strain tensors.

1. We shall denote the principal stresses by σ_1 and σ_2 ($\sigma_i > 0$ corresponds to compression). Then the values of the tangential τ_n and normal σ_n stress components on the area inclined to the 1-axis at the angle α are

$$\tau_n = \frac{\sigma_1 - \sigma_2}{2} \sin 2\alpha, \qquad \sigma_n = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2\alpha \tag{1.1}$$

and we have

$$d\tau_n \,/\, d\sigma_n = \operatorname{ctg} 2\,|\alpha|\, \operatorname{sgn}\alpha \tag{1.2}$$

We shall assume that the considered macro-point of the medium is in its critical equilibrium state, provided that one of the elementary areas (the critical equilibrium area) passing through this point is such, that the stresses acting on it are subject to the generalized Coulomb's condition, i.e. they are balanced by the reaction force, the latter being the dry friction force R acting, in general, at the angle β to the area in question

$$\tau_n \cos\beta - \sigma_n \sin\beta = R, \qquad \tau_n \sin\beta + \sigma_n \cos\beta = N$$

$$R = N\theta \operatorname{tg}\Phi_0 + c_0\theta, \qquad \theta = \operatorname{sgn} R, \qquad N > 0 \tag{1.3}$$

where Φ_0 is the true angle of internal friction and c_0 denotes true cohesion.

Conditions (1.3) can be reduced to

$$\tau_n' = \sigma_n \operatorname{tg}(\beta + \theta\Phi_0) + c_0 \frac{\theta \cos\Phi_0}{\cos(\beta + \theta\Phi_0)} \tag{1.4}$$

and we have

$$\frac{d\tau_n}{d\sigma_n} = \operatorname{tg}(\beta + \theta\Phi_0) \tag{1.5}$$

Comparing (1.2) and (1.5) we find the angle α of inclination of the area (line) of critical equilibrium as the solution of

$$\operatorname{ctg} 2\,|\alpha|\, \operatorname{sgn}\alpha = \operatorname{tg}(\beta + \theta\Phi_0)$$

i. e.

$$|\alpha| = {}^1/_4\pi - {}^1/_2(\beta + \theta\Phi_0)\operatorname{sgn}\alpha \qquad (1.6)$$

2. We shall further assume that the force R is collinear with a certain direction, which makes the angle λ with the 1-axis. Then $\beta = (|\lambda| - |\alpha|)\operatorname{sgn}\alpha$ (we measure α and λ in the same direction), and from (1.6) we have

$$|\alpha| = {}^1/_4\pi + {}^1/_2(|\alpha| - |\lambda| - \theta\Phi_0\operatorname{sgn}\alpha)$$

which in turn yields

$$|\alpha| + |\lambda| + \Phi_0 = {}^1/_2\pi, \qquad \theta\operatorname{sgn}\alpha = 1$$

$$|\alpha| + |\lambda| - \Phi_0 = {}^1/_2\pi, \qquad \theta\operatorname{sgn}\alpha = -1 \qquad (2.1)$$

We shall now assume that the force R is collinear with the tangent, at the point considered, to the slip line (*) (lines of the maximum shear rate $\gamma_{ij} = \varepsilon_{ij},\ i \neq j$)

$$\operatorname{tg} 2\lambda = \frac{\varepsilon_{11} - \varepsilon_{22}}{2\varepsilon_{12}} \qquad (2.2)$$

and, that it is of the same sign as the rate of shear γ along the slip line $\theta = \operatorname{sgn}\gamma_{max}$.

If the stress and rate of strain tensors are collinear (the hypothesis of [3]), then

$$|\lambda| = {}^1/_4\pi, \quad |\alpha| = {}^1/_4\pi - \Phi_0, \qquad \beta = \Phi_0\operatorname{sgn}\alpha, \qquad \theta\operatorname{sgn}\alpha = 1 \qquad (2.3)$$

or

$$|\lambda| = {}^1/_4\pi, \quad |\alpha| = {}^1/_4\pi + \Phi_0, \qquad \beta = -\Phi_0\operatorname{sgn}\alpha, \qquad \theta\operatorname{sgn}\alpha = -1$$

We can easily see that the condition

$$\theta\operatorname{sgn}\alpha = 1 \qquad [\operatorname{sgn}\gamma_{max} = \operatorname{sgn}\alpha, \qquad |\gamma_{max}| = {}^1/_2|\varepsilon_1 - \varepsilon_2|]$$

corresponds to the inequality $\varepsilon_1 > \varepsilon_2$, the maximum (in modulo) stresses and principal strain rates coincide.

In this case we have, for an incompressible medium ($\varepsilon_1 + \varepsilon_2 = 0$), the following expression for the dissipation of mechanical work per unit (elementary) volume

$$W^{\cdot} = \sigma_1\varepsilon_1 + \sigma_2\varepsilon_2 = \frac{(\sigma_1 - \sigma_2)(\varepsilon_1 - \varepsilon_2)}{2} = 2A^{\cdot} \qquad \left(A^{\cdot} = \frac{\sigma_1 - \sigma_2}{2}\frac{\varepsilon_1 - \varepsilon_2}{2}\right)$$

where $A^{\cdot}$ denotes the dissipation on each (of the two) slip area.

If $\theta\operatorname{sgn}\alpha = -1$, then the minimum principal strain rate is directed along the axis of the maximum principal stress and this would imply $W^{\cdot} = 0$, therefore we choose the condition $\theta\operatorname{sgn}\alpha = 1$ which leads to the condition of collinearity in any coordinate system

$$\frac{\sigma_{11} - \sigma_{22}}{2\sigma_{12}} = \frac{\varepsilon_{11} - \varepsilon_{22}}{\varepsilon_{12}} \qquad (2.4)$$

This, in turn, implies that, by the virtue of the incompressibility of the medium ($\varepsilon_{11} + \varepsilon_{22} = 0$), the tensor-deviators of the stresses and strain rates

$$\varepsilon_{11} = \zeta^{\cdot}\left(\sigma_{11} - \frac{\sigma_{11} + \sigma_{22}}{2}\right), \qquad \varepsilon_{22} = \zeta^{\cdot}\left(\sigma_{22} - \frac{\sigma_{11} + \sigma_{22}}{2}\right)$$

$$ {}^1/_2\varepsilon_{12} = \zeta^{\cdot}\sigma_{12}, \qquad \zeta^{\cdot} \geqslant 0 \qquad (2.5)$$

are coaxial and similar.

When (2.3) holds then, naturally, the critical equilibrium relation (1.4) is reduced to the usual form of the Coulomb's condition

$$|\tau_n| = \sigma_n \operatorname{tg}\Phi + c, \qquad \Phi = 2\Phi_0, \qquad c = c_0\frac{\cos\Phi_0}{\cos 2\Phi_0} \qquad (2.6)$$

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*) Our procedure differs from the usual in that here it is essential to obtain the slip lines (lines of the maximum shear rate) and the critical equilibrium lines (see the condition (1.3)).